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A PROPOSED COMPUTATIONAL METHOD FOR ESTIMATION OF ORBITAL ELEMENTS, DRAG COEFFICIENTS AND POTENTIAL FIELD PARAMETERS FROM SATELLITE MEASUREMENTS–I: INTRODUCTION

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PREFACE

As part of The RAND Corporation's continuing study of system identification, this Memorandum presents a method of applying quasi-linearization and digital computers to the problem of simultaneously estimating the density of the earth's atmosphere at high altitudes and the geopotential, making use of measurements of a satellite's motion through the atmosphere. This study will be of interest to those working in the fields of space science and meteorology.

SUMMARY

A new approach to the problem of estimating the density of the earth's atmosphere at high altitudes is presented. Using measurements of a satellite's motion through the atmosphere, we formulate the problem as a nonlinear multi-point boundary value problem which can be solved using quasilinearization and high-speed computers. A general machine program has been written, and the results of some illustrative preliminary tests are presented. This Memorandum is an introduction to a more detailed study.

CONTENTS

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I. INTRODUCTION

In a recent paper King-Hele[1] has described current views of
the problem of estimating the density of the earth's atmosphere at
altitudes between 200 and 1000 km employing measurements of earth
satellites. In the section on future developments, he mentions the
use of measurements made during a single revolution to gain improved
resolution. The aim of this Memorandum is to show how measurements
of displacement and velocity can be converted into estimates of or-
bital elements, drag coefficients and potentials. We shall employ a
quasilinearization technique.[2] Earlier applications to orbit deter-
mination are given in Refs. 3 and 4. Our procedure takes full advan-
tage of the modern computer's ability to integrate large systems of
nonlinear differential equations subject to multi-point boundary
conditions.

II. METHOD

We shall present our method through the discussion of a simpli-
fied one-dimensional example so as not to obscure the trend of
thought in a jungle of equations. Consider the motion of a mass des-
cribed by the differential equations

$$\dot{x} = v \tag{1}$$

$$\dot{v} = g - kv^2 . \tag{2}$$

The parameter g is associated with the gravitational field, and the
parameter k is a drag coefficient. We suppose that at various times
t_i, i = 1, 2, ..., N, noisy observations are made of the displacement
and velocities, the measured values at time t_i being b_i and c_i,
i = 1, 2, ..., N. We wish to estimate the parameters g and k and the
initial conditions (orbital elements), x(0) and v(0), so that we shall
minimize S, the sum of the squares of the deviations between the ob-
served values and the theoretical values,

$$S = \sum_{i=1}^{N} \left\{ \lambda_1 \left[x(t_i) - b_i \right]^2 + \lambda_2 \left[v(t_i) - c_i \right]^2 \right\} , \tag{3}$$

where λ_1 and λ_2 are certain weights. The theoretical values $x(t_i)$
and $v(t_i)$ are obtained by numerically solving the differential equa-
tions (1) and (2) with the considered values of g and k and the con-
sidered initial conditions. The values of these four quantities that
minimize S are our estimates.

We solve this minimization problem by a numerical technique which has worked well in many instances in the past.[2] The scheme is a successive approximation technique. We denote the previous approximation of the functions x and v by x^0 and v^0, for $t_1 \le t \le t_N$, and the new ones by x^1 and v^1. The previous estimates of g and k are denoted by g^0 and k^0, and the new ones by g^1 and k^1. The linearized equations for x^1 and v^1 are

$$\dot{x}^1 = v^0 + (v^1 - v^0) = v^1 \tag{4}$$

$$\dot{v}^1 = g^0 - k^0(v^0)^2 + (-2k^0 v^0)(v^1 - v^0) + (g^1 - g^0) \tag{5}$$
$$+ (-v^0)^2 (k^1 - k^0) .$$

These equations are obtained by expanding the right-hand sides of Eqs. (1) and (2) about $x = x^0$, $v = v^0$, $k = k^0$ and $g = g^0$ and retaining only the terms that are linear in x^1, v^1, k^1 and g^1. We wish to determine the functions x^1 and v^1 and the constants g^1 and k^1 so as to minimize the sum S_1,

$$S_1 = \sum_{i=1}^{N} \left\{ \lambda_1 \left[x^1(t_i) - b_i \right]^2 + \lambda_2 \left[v^1(t_i) - c_i \right]^2 \right\} . \tag{6}$$

This problem may be solved computationally at the expense of solving a few initial value problems by the method of complementary functions, a task for which modern computers are admirably suited.

First we produce numerically the solutions of the homogeneous equations

$$\dot{h}_{11} = h_{21}, \tag{7}$$

$$\dot{h}_{21} = (-2k^0 v^0) \, h_{21}, \qquad t_1 \le t \le t_N, \tag{8}$$

$$h_{11}(t_1) = 1, \tag{9}$$

$$h_{21}(t_1) = 0, \tag{10}$$

and

$$\dot{h}_{12} = h_{22}, \tag{11}$$

$$\dot{h}_{22} = (-2k^0 v^0) \, h_{22}, \qquad t_1 \le t \le t_N, \tag{12}$$

$$h_{12}(t_1) = 0, \tag{13}$$

$$h_{22}(t_1) = 1. \tag{14}$$

Then we numerically produce the particular solutions p_1 and p_2

$$\dot{p}_1 = p_2, \tag{15}$$

$$\dot{p}_2 = g^0 - k^0 (v^0)^2 - (2k^0 v^0) \, (p_2 - v^0) - g^0 + (-v^0)^2 \, (-k^0), \tag{16}$$

on the interval $t_1 \le t \le t_N$ using the initial conditions

$$p_1(t_1) = 0, \tag{17}$$

$$p_2(t_1) = 0. \tag{18}$$

And finally we produce numerically the particular solutions

$$\mathring{q}_1 = q_2, \qquad\qquad q_1(t_1) = 0, \qquad\qquad (19)$$

$$\mathring{q}_2 = (-2k^0 v^0)\, q_2 + 1, \qquad q_2(t_1) = 0, \qquad\qquad (20)$$

and

$$\dot{r}_1 = r_2, \qquad\qquad r_1(t_1) = 0, \qquad\qquad (21)$$

$$\dot{r}_2 = (-2k^0 v^0)\, r_2 + (-v^0)^2, \qquad r_2(t_1) = 0,$$

all for $t_1 \le t \le t_2$.

The general solution to Eqs. (4) and (5) may now be written in the form

$$x^1 = \alpha\, h_{11} + \beta\, h_{12} + p_1 + g^1\, q_1 + k^1\, r_1 \qquad\qquad (22)$$

$$v^1 = \alpha\, h_{21} + \beta\, h_{22} + p_2 + g^1\, q_2 + k^1\, r_2 \,. \qquad\qquad (23)$$

The constants α and β are the new estimates of the orbital elements:

$$x^1(t_1) = \alpha \qquad\qquad (24)$$

$$v^1(t_1) = \beta \,. \qquad\qquad (25)$$

To find the values of the parameters α, β, g^1 and k^1 which minimize the sum of the squares of the deviations S_1 we return to Eq. (6) and replace $x^1(t_i)$ and $v^1(t_i)$ by their equivalents from Eqs. (22) and (23). In this way we see that S is quadratic in the parameters α, β,

g^1 and k^1. We find the values of these parameters which minimize S_1 by solving the linear system of algebraic equations

$$\frac{\partial S_1}{\partial \alpha} = 0 \tag{26}$$

$$\frac{\partial S_1}{\partial \beta} = 0 \tag{27}$$

$$\frac{\partial S_1}{\partial g^1} = 0 \tag{28}$$

$$\frac{\partial S_1}{\partial k^1} = 0 . \tag{29}$$

Once the minimizing values of α, β, g^1 and k^1 have been determined, we return to Eqs. (22) and (23) and produce the numerical values of the functions x^1 and v^1 on the interval $t_1 \le t \le t_N$. This completes a basic cycle of the iterative calculation. The rapid quadratic convergence which can be expected is discussed in Ref. 2.

The initial approximations for the constants g and k and the functions x and v (for $t_1 \le t \le t_N$) may be obtained, e.g., by using the best estimates of g and k available and then integrating Eqs. (1) and (2) from t_1 to t_N, using the best available estimates of $x(0)$ and $v(0)$.

III. SOME NUMERICAL EXPERIMENTS

First we present the results of an experiment designed to show the rapidity of convergence of the technique. We integrated Eqs. (1) and (2) using

$$g = 32.2 \tag{30}$$

$$k = 32.2 \times 10^{-3} = 0.0322 \tag{31}$$

and

$$x(0) = 0.0 \tag{32}$$

$$v(0) = 0.0 \;, \tag{33}$$

the integration extending from $t = 0$ to $t = 1.0$. The results are given in columns 3 and 5 of Table 1, and provide us with a set of displacement and a set of velocity measurements for $t = .1, .2, \ldots, 1.0$. The estimation problem consists of assuming these twenty observations as given and seeking to uncover the values of the parameters g and k as well as the initial conditions $x(0)$ and $v(0)$.

We employed the method sketched in Section II, assuming our initial estimates of the unknowns to be

$$x^0(0) = 0.1 \tag{34}$$

$$v^0(0) = -0.1 \tag{35}$$

$$g^0 = 30.0 \tag{36}$$

$$k^0 = 0.003 \;. \tag{37}$$

Table 1

OBSERVED AND THEORETICAL VALUES OF DISPLACEMENTS AND VELOCITIES

Time	x	x Observations b_i	v	v Observations c_i
0.1	0.16072253E 00	0.16072254E 00	0.32089146E 01	0.32089171E 01
0.2	0.63959687E 00	0.63959709E 00	0.63524209E 01	0.63524224E 01
0.3	0.14270092E 01	0.14270096E 01	0.93703303E 01	0.93703309E 01
0.4	0.25077758E 01	0.25077762E 01	0.12212048E 02	0.12212049E 02
0.5	0.38622596E 01	0.38622599E 01	0.14839446E 02	0.14839446E 02
0.6	0.54676883E 01	0.54676887E 01	0.17227990E 02	0.17227991E 02
0.7	0.72995026E 01	0.72995033E 01	0.19366275E 02	0.19366278E 02
0.8	0.93325923E 01	0.93325937E 01	0.21254341E 02	0.21254349E 02
0.9	0.11542327E 02	0.11542329E 02	0.22901304E 02	0.22901317E 02
1.0	0.13905332E 02	0.13905336E 02	0.24322771E 02	0.24322791E 02

The E notation refers to exponents of 10, i.e., 0.32089171E 01 = 3.2089171.

Notice that the initial approximation of k is incorrect by a factor greater than ten. Our initial estimates of the functions x(t) and v(t) were obtained by integrating Eqs. (1) and (2) subject to the conditions in Eqs. (34) through (37). The results are displayed in Table 2. They show that excellent estimates are obtained by the third approximation. The second and fourth columns of Table 1 contain the values of the displacements and velocities obtained using those estimates in Eqs. (1) and (2).

Now we turn to the results of an experiment involving noisy observations. The values of the velocities used in the earlier experiment were corrupted by adding about two percent gaussian noise to each observation. These new noisy observations are listed in column 5 of Table 3. The displacement observations are the same as before, and the value of g is kept at 32.2. We found the minimizing values of the unknowns to be

$$x(0) = 0.928198 \times 10^{-1} \tag{38}$$

$$v(0) = -0.270570 \tag{39}$$

$$k = 0.298726 \times 10^{-1} . \tag{40}$$

This serves as a warning by showing that quite small errors in the observations can lead to significantly larger errors in the estimates. The value of using numerical experiments such as these to aid in estimating the accuracy and number of the measurements required to obtain the desired accuracy in the estimates is evident.

Results such as these are obtained in an execution time of a few seconds on an IBM 7044.

Table 2

RESULTS OF SUCCESSIVE APPROXIMATIONS

	Zero	First	Second	Third	True Value
$x(0)$	0.100000	0.193715×10^{-6}	-0.571534×10^{-4}	0.536442×10^{-6}	0.000000
$v(0)$	-0.100000×10^2	0.119209×10^{-6}	0.128591×10^{-2}	-0.200272×10^{-4}	0.000000
g	0.300000×10^2	0.321997×10^2	0.321992×10^2	0.322000×10^2	0.322000×10^2
k	0.300000×10^{-2}	0.321995×10^{-1}	0.320768×10^{-1}	0.322000×10^{-1}	0.322000×10^{-1}

Table 3

THE FIT TO THE NOISY OBSERVATIONS

Time	x	x Observations b_i	v	v Observations c_i
0.1	0.226581E 00	0.160722E 00	0.294152E 01	0.320540E 01
0.2	0.679259E 00	0.639597E 00	0.609782E 01	0.627940E 01
0.3	0.144238E 01	0.142701E 01	0.914155E 01	0.882816E 01
0.4	0.250218E 01	0.250778E 01	0.120240E 02	0.121049E 02
0.5	0.384052E 01	0.386226E 01	0.147071E 02	0.144243E 02
0.6	0.543608E 01	0.546769E 01	0.171651E 02	0.170832E 02
0.7	0.726556E 01	0.729950E 01	0.193840E 02	0.192415E 02
0.8	0.930481E 01	0.933259E 01	0.213608E 02	0.212337E 02
0.9	0.115298E 02	0.115423E 02	0.231011E 02	0.236843E 02
1.0	0.139176E 02	0.139053E 02	0.246175E 02	0.243658E 02

IV. DISCUSSION

A general program in FORTRAN IV has been written which solves nonlinear multi-point boundary value problems using the method sketched in Section II. It is available from the authors. We plan to investigate its utility in estimating parameters of the geopotential and drag coefficient from satellite measurements.

Previous experience plus experiments similar to those reported here show that small parameters in differential equations can be estimated successfully using experimental observations on the solutions.

It must not be thought, however, that matters are completely routine. Various problems such as instability of the linearized equations, limited memory, and ill-conditioning of the linear algebraic equations can and do arise. Methods for overcoming these difficulties are given in Refs. 2 and 5.

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10. ABSTRACT

'A new computational approach for estimating the density of the earth's atmosphere at high altitudes, based on measurements of a satellite's motion through the atmosphere. The problem is formulated as a nonlinear multipoint boundary-value problem that can be solved using quasilinearization and high-speed digital computers. A general machine program has been written in FORTRAN IV, and the results of some illustrative preliminary tests are presented.

11. KEY WORDS

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